SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH 2-POINT AND INTEGRAL BOUNDARY CONDITIONS

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THESIS

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ABSTRACT

In sophomore and junior level ordinary differential equations one studies the classical Sturm-Liouville boundary value problem, where the boundary conditions are of the separated type. It is well known that under very reasonable hypotheses this problem has a discrete set of non-trivial solutions for a discrete set of eigenvalues which are countably infinite and tend to infinity. It is the purpose of this thesis to study the question of whether similar results hold for problems when the boundary conditions are replaced by conditions of the non-separated type and also conditions where an integral is added. In doing so, we are able to generalize some recent results of Etgen and Tefteller.

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I. INTRODUCTION

A. HISTORY

In the study of ordinary differential equations the existence and uniqueness of solutions, given initial conditions, is well known. However, for boundary value problems existence and uniqueness theorems are neither obvious nor easy. The question is also posed that if solutions do exist, what is the character of these solutions and how does this character change when the differential equation or boundary conditions change.

In fact, one of the most important questions is non-uniqueness - the existence of non-trivial solutions for the homogeneous equation. Consider the system

$$L(u) = a_0(x)\, u'' + a_1(x)\, u' + a_2(x)\, u = f(x) \qquad a < x < b$$

where a_0 , a_1 , a_2 are continuous and $f(x)$ is piecewise continuous, subject to the boundary conditions

$$\begin{aligned} B_1(u) &= \alpha_{11}u'(a) + \alpha_{12}u(a) + \alpha_{13}u'(b) + \alpha_{14}u(b) = 0 \\ B_2(u) &= \alpha_{21}u'(a) + \alpha_{22}u(a) + \alpha_{23}u'(b) + \alpha_{24}u(b) = 0, \end{aligned} \tag{1}$$

and the system

$$\begin{aligned} L(u) &= 0 \qquad & a < x < b \\ B_1(u) &= 0 \qquad & B_2(u) = 0. \end{aligned} \tag{2}$$

Any two solutions of (1) differ by a solution of (2). If (2) has only the trivial solution, then (1) has at most one solution. In the case that (2) has a non-trivial solution, (1) either has no solution or has

many solutions. Also, if u_1 is a solution of (1) corresponding to f_1 and u_2 is a solution of (1) corresponding to f_2, then $c_1u_1 + c_2u_2$ is a solution corresponding to $c_1f_1 + c_2f_2$. Finally, if (2) has only the trivial solution, then the Green's function for (1) exists and is unique.

The study of boundary value problems for linear second order differential equations dates from the time of Euler and D'Alembert. The first somewhat general theory of such problems, however, is that given by Sturm [14] in 1836. Sturm's paper is concerned with the differential equation

$$(K(x,\lambda)u')' - G(x,\lambda)u = 0$$

with two-point boundary conditions of the form

$$\alpha_1(\lambda)u(x_1) - \beta_1(\lambda)K(x_1,\lambda)u'(x_1) = 0$$

$$\alpha_2(\lambda)u(x_2) + \beta_2(\lambda)K(x_2,\lambda)u'(x_2) = 0 .$$

Under certain general conditions which include the assumption that $K(x,\lambda)$, $G(x,\lambda)$, $\alpha_1(\lambda)/\beta_1(\lambda)$ and $\alpha_2(\lambda)/\beta_2(\lambda)$ are monotone decreasing functions of λ, it is established that there exist infinitely many eigenvalues, $\lambda_1 < \lambda_2 < \ldots$, and that the eigenfunction u_n corresponding to λ_n has exactly $n - 1$ zeros on $x_1 < x < x_2$. Sturm also proved that given two equations

$$u''(x) + F(x)u(x) = 0 \qquad \text{(i)}$$

$$u''(x) + G(x)u(x) = 0 \qquad \text{(ii)}$$

where F and G are positive, continuous and $G(x) \geq F(x)$ in (a,b), if (i) has a solution $u(x)$ having two consecutive zeros at $x = x_1$, $x = x_2$, $a < x_1 < x_2 < b$ and $v(x)$ is a solution of (ii) having $v(x_1) = 0$, then $v(x)$ has at least one zero x_3, $x_1 < x_3 < x_2$.

Porter [12], in 1902, solved a system of boundary value problems using a rigorous passage to the limit from a system of difference equations to a differential system. This method was originally used by Sturm, but was never published. Other authors including Bôcher, Courant and Whyburn have used this method to solve various boundary value problems. However, W. T. Reid [13] remarks that most of the results obtained by this method have been later proven by methods "more elegant in detail."

In 1908, G. D. Birkhoff [1] developed asymptotic expressions for the solutions of a single equation of the n^{th} order involving a parameter and applied his results to a boundary value problem which is linear in the characteristic parameter λ and involves two-point boundary conditions. The coefficients of the differential equation and the boundary conditions are not assumed to be real; and the system is not supposed to be self-adjoint. In certain general cases Birkhoff obtains the existence of infinitely many eigenvalues. He also proved existence and oscillation theorems for a second order linear differential equation with self-adjoint boundary conditions.

Hilbert [9], in 1912, proved that a boundary value problem consisting of a single second order differential equation with real coefficients and corresponding self-adjoint boundary conditions is equivalent to an integral equation with a real symmetric closed kernel. Then using his theory of such integral equations, he proved that the given boundary value problem has an infinite number of real and no complex eigenvalues.

In 1914, Lichtenstein [11] considered boundary value problems involving a single second order linear differential equation in the characteristic parameter with associated Sturmian boundary conditions. By expanding admissible functions $n(x)$ in Fourier series, he showed

that the eigenvalues of the given problem were identical with the characteristic values of an infinite system of linear algebraic equations

$$(\delta_{ij} + \lambda K_{ij})x_j = 0 \qquad i,j = 1,2, \ldots$$

in the variables x_i . Under certain general conditions he proved the existence of infinitely many eigenvalues for the given problem. Anna Pell Wheeler, in 1927, applied the theory of linear algebraic equations in an infinite number of variables to a second order differential equation with Sturmian boundary conditions.

What may be considered the next step is studying n^{th} order linear systems with boundary conditions at a finite number k of points of the interval. The boundary conditions are linear combinations of the values of the solution and its first $(n-1)$ derivatives at the k points. C. E. Wilder, in 1917, was the first to systematically study such problems. He worked with the single n^{th} order equation with conditions at k points. He also defined a Green's function for his system and investigated the adjoint relationship of the system. In 1948, G. J. Haltiner [8] studied second order linear differential systems using a new definition of the adjoint due to R. E. Langer [10]. He also introduced a very interesting representation of the Green's function based on the eigenfunctions of the equation and its adjoint. In 1928, Whyburn [17] proved existence and oscillation theorems for second order systems containing a parameter and with boundary conditions one of which was at k points. His results differed from those usually obtained in that the eigenvalues were not isolated, but occurred in sets or bands. This is significant because of its application to spectral analysis.

Several authors, most notably W. T. Reid in 1935, studied the relationship between k point boundary value problems and the calculus of variations. The discussion hinges on the relation between the Jacobi necessary condition in calculus of variations and second order differential equations with boundary conditions at two points. It extends to the k point problem. Reid effectively reduced the k point case to one that involves $k-1$ systems each of the two point type.

The boundary conditions may be further complicated by adding an integral term. The earliest apparent study of this type was by Picone who, in 1908, studied relationships existing between certain special n^{th} order differential equations with integral boundary conditions and integral equations.

In 1912, due to its relevance to certain hydrodynamics problems, von Mises [16] studied the second order system

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + (\lambda q(x)-r(x))u = 0$$

$$\int_a^b A(x)u(x)dx = 0 \qquad \int_a^b B(x)u(x)dx = 0$$

where p , q , r , A , B are continuous functions of x and λ is a parameter. He obtained existence and uniqueness theorems for this system via an application of Sturm's method of passage to the limit from an algebraic system.

Bôcher, in the same year, postulated a transformation for second order systems which would replace the integral conditions by conditions at two points. However, the transformation proved itself impractical due to its dependence upon a solution to an altered form of the original equation.

The next step, by Tamarkin [15] was to study systems of n^{th} order with k point and integral conditions.

Whyburn, in 1928, proved existence and oscillation theorems for linear second order systems with one integral boundary condition and one two-point boundary condition. The coefficients of the system contained a parameter in a general fashion. As in his study of k point conditions, the eigenvalues were not isolated but occurred in sets. In 1972, Etgen and Tefteller [6] extended the work of Whyburn to the case where one boundary condition is a two point type and the other is at two points plus an integral. They also proved that the eigenvalues occurred in sets or bands.

It is possible to propose even more additions and/or restrictions to the boundary conditions. However, in order to obtain a more complete understanding of the theory, it is valuable to study equations of a specific order with boundary conditions of a specific type. Consequently, this thesis examines second order linear differential equations with several types of boundary conditions.

B. PURPOSE OF THE THESIS

In sophomore and junior level ordinary differential equations one studies the classical Sturm-Liouville boundary value problem,

$$\frac{d}{dx}[p(x)\frac{du}{dx}] - q(x)u(x) = \lambda r(x)u(x) \tag{3}$$

where the boundary conditions are all of the separated type,

$$\begin{aligned} \alpha_{11}y'(a) - \alpha_{12}\,y(a) &= 0 \\ \alpha_{23}y'(b) - \alpha_{24}\,y(b) &= 0 \end{aligned} \tag{4}$$

It is well known that under very reasonable hypotheses equation (3) with conditions (4) has a discrete set of non-trivial solutions for a discrete set of eigenvalues which are countably infinite and tend to infinity. For a proof see [3]. It is the purpose of this thesis to study the question of whether similar results hold for problems when the boundary conditions (4) are replaced by non-separated conditions of the form

$$\begin{aligned} \alpha_{11}y'(a) + \alpha_{12}y(a) + \alpha_{13}y'(b) + \alpha_{14}y(b) &= 0 \\ \alpha_{21}y'(a) + \alpha_{22}y(a) + \alpha_{23}y'(b) + \alpha_{24}y(b) &= 0 \end{aligned} \tag{5}$$

and also conditions where an integral term is added to equations (5). We consider systems of two first order equations because in most cases the extension to n equations is then apparent.

The material is presented in such a way as to show the logical progression from boundary conditions at two points to the more complex conditions involving an integral term. We show that the work of Etgen and Tefteller is too restrictive. By careful analysis of their work we show that their major theorems can all be combined and that together with some of the previous work can be considered a special case of a theorem we prove.

C. OUTLINE OF THE CHAPTERS

In Chapter II the existence and oscillation theorems of G. D. Birkhoff are studied. The equation is of the type (3) with $p(x) \equiv 1$, but the boundary conditions are of the form (5). Relying on the solutions to Sturm's problem (3), (4), the eigenvalues are generated by appealing to the fact that a certain continuous solution condition changes sign

and hence must vanish as desired. The oscillation theorem follows by observing the behavior of the boundary conditions at the endpoints and also for large negative λ .

Chapter III carries the discussion to the next logical question. Since the solutions exist, how may they be found and characterized? G. J. Haltiner studied coupled boundary condition problems of the form (3) and (5) from the point of view of adjoints and Green's functions. He employed a novel definition of the adjoint due to Langer and has obtained an interesting way of writing a Green's function, which we illustrate with a simple example. This Green's function can be used to solve non-homogeneous equations when λ is not an eigenvalue.

Boundary conditions with an integral term are then introduced in Chapter IV. The proof of the existence theorem relies on the fact that the zeros of a known problem can be used to isolate the zeros of the problem under consideration. Similarly, in the oscillation theorem its proof rests on the ability to separate the zeros of the two problems.

Chapter V progresses to boundary conditions of a more general type, but it is shown that the type studied in Chapter IV is in fact a special case of this general type. In contrast, however, the method of proof is different. Proof of the existence theorem uses the Prüfer substitution to equate the solutions to trigonometric functions whose zeros are easily studied. The argument is of a geometric nature and does not require use of the principle of superposition. Thus, it can be and has been used for second order equations of the non-linear type. However, we see no way of extending this method of proof to anything higher than second order equations.

II. THE PROBLEM OF G. D. BIRKHOFF

A. FORMULATION OF THE PROBLEM

Consider the second order linear differential equation

$$L(u,\lambda) = u'' + q(x,\lambda)u = 0 \qquad a \le x \le b \tag{1.1}$$

with the boundary conditions

$$\alpha_{11}u'(a) + \alpha_{12}u(a) = \alpha_{13}u'(b) + \alpha_{14}u(b) \tag{1.2a}$$

$$\alpha_{21}u'(a) + \alpha_{22}u(a) = \alpha_{23}u'(b) + \alpha_{24}u(b) \tag{1.2b}$$

If we require that

$$\alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21} = \alpha_{13}\,\alpha_{24} - \alpha_{14}\,\alpha_{23} \tag{1.3}$$

where α_{11} , α_{12} , α_{13} , α_{14} and α_{21} , α_{22} , α_{23} , α_{24} are real but not proportional, then the problem is said to be self-adjoint.

Our first concern is to develop a theorem which guarantees the existence of non-trivial solutions for the problem (1.1), (1.2). Birkhoff [2] has proved such a theorem and also one on the nature of the oscillation of the real solutions.

For this theorem, we make two assumptions on $q(x,\lambda)$:

1. $q(x,\lambda)$ is continuous in x and λ for all real λ provided x is in the interval (a,b).

2. $q(x,\lambda)$ increases as λ increases, satisfying

$$\lim_{\lambda\to -\infty} q(x,\lambda) = -\infty \qquad \lim_{\lambda\to +\infty} q(x,\lambda) = +\infty \tag{1.4}$$

We are interested in the case where (1.3) is not zero. Furthermore, since we can change α_{11} , α_{12} , α_{13} , α_{14} or α_{21} , α_{22} , α_{23} , α_{24} by a constant factor, we may assume

$$\alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21} = \alpha_{13}\,\alpha_{24} - \alpha_{14}\,\alpha_{23} = 1 \tag{1.5}$$

For convenience of notation, we write

$$\begin{aligned} L_0[u(x)] &\equiv \alpha_{11}u'(x) + \alpha_{12}u(x) \qquad & M_0[u(x)] &\equiv \alpha_{13}u'(x) + \alpha_{14}u(x) \\ L_1[u(x)] &\equiv \alpha_{21}u'(x) + \alpha_{22}u(x) \qquad & M_1[u(x)] &\equiv \alpha_{23}u'(x) + \alpha_{24}u(x) \end{aligned} \tag{1.6}$$

Equation (1.1) with (1.2) is a boundary value problem having the obvious solution $u \equiv 0$. The existence of a non-trivial solution is not at all obvious. However, we do know that there exist solutions to the initial value problem. Thus, Birkhoff's procedure is to take a particular solution to an initial value problem and show that the boundary value problem can be solved so as to obtain a non-trivial solution. Following Birkhoff, we find solutions $u_0(x,\lambda)$ and $u_1(x,\lambda)$ of (1.1) satisfying

$$\begin{aligned} L_0[u_0(a,\lambda)] &= 0 \qquad & L_0[u_1(a,\lambda)] &= 1 \\ L_1[u_0(a,\lambda)] &= 1 \qquad & L_1[u_1(a,\lambda)] &= 0 \end{aligned} \tag{1.7}$$

From these equations we can easily show

$$L_0[u_0(a,\lambda)]\, L_1[u_1(a,\lambda)] - L_0[u_1(a,\lambda)]\, L_1[u_0(a,\lambda)] = -1 \tag{1.8}$$

$$M_0[u_0(b,\lambda)]\, M_1[u_1(b,\lambda)] - M_0[u_1(b,\lambda)]\, M_1[u_0(b,\lambda)] = -1 \tag{1.9}$$

Now, by (1.7) we have fixed u_i , u_i' , $i = 0$, 1 at $x = a$ and thus determined $u_i(x)$, $i = 0,1$, $a \le x \le b$.

By (1.7), we also know that u_0 , u_1 are linearly independent and thus any solution u of (1.1) is of the form

$$u = c_0 u_0 + c_1 u_1 \tag{1.10}$$

Substituting (1.10) into (1.2) and simplifying by (1.7), we obtain the following two equations which are linear in c_0 and c_1 :

$$c_1[1 - M_0\ [u_1(b,\lambda)]] - c_0 M_0[u_0(b,\lambda)] = 0 \tag{1.11}$$

$$c_0[1 - M_1\ [u_0(b,\lambda)]] - c_1 M_1[u_1(b,\lambda)] = 0 \tag{1.12}$$

Then a non-zero solution of (1.1), (1.2) will exist if and only if $\varphi(\lambda) = 0$ where

$$\varphi(\lambda) = \begin{vmatrix} -M_0[u_0(b,\lambda)] & 1 - M_0[u_1(b,\lambda)] \\ 1 - M_1[u_0(b,\lambda)] & -M_1[u_1(b,\lambda)] \end{vmatrix} \tag{1.13}$$

Setting $x = b$ in (1.9), (1.13) becomes

$$\varphi(\lambda) \equiv M_0[u_1(b,\lambda)] + M_1[u_0(b,\lambda)] - 2 \tag{1.14}$$

Thus the necessary and sufficient condition that there exists a non-zero solution when $\lambda = \bar{\lambda}$ is that $\varphi(\bar{\lambda}) = 0$.

B. THE EXISTENCE THEOREM

In 1836, Sturm considered the case where (1.3) vanished, so that (1.1) holds but the boundary conditions (1.2) are replaced by

$$\alpha_{11}u'(a) + \alpha_{12}u(a) = 0 \qquad \alpha_{23}u'(b) + \alpha_{24}u(b) = 0 \tag{1.2c}$$

and thus the boundary conditions are uncoupled. It is interesting to note that (1.2c) is, in our notation, $L_0[u(a)] = 0$ and $M_1[u(b)] = 0$;

and thus Birkhoff's u_0 will, for certain values of λ , become Sturm's. Sturm proved the existence of an infinite set of eigenfunctions and eigenvalues, which we now use. Assume we know them and order them

$$\bar{\lambda}_1, \bar{\lambda}_2, \ldots \qquad (\bar{\lambda}_1 < \bar{\lambda}_2 < \ldots)$$

We can then separate the λ - axis into intervals

$$(-\infty, \bar{\lambda}_1), (\bar{\lambda}_1, \bar{\lambda}_2), (\bar{\lambda}_2, \bar{\lambda}_3), \ldots$$

We note that the intervals are not unique because conditions (1.2) can be replaced by any two linearly independent conditions resulting from a linear combination. Then, the existence theorem as stated by Birkhoff [2] is

Theorem 1.1:

There exists an infinite set of values $\lambda_1, \lambda_2, \ldots$ of λ furnishing solutions of (1.1), (1.2). Furthermore, taking the quantities in order of increasing magnitude counting each double value twice, there are the following possible cases:

Ia $\bar{\lambda}_1 < \lambda_1 \le \bar{\lambda}_2 \le \lambda_2 < \bar{\lambda}_3 < \lambda_3 \le \bar{\lambda}_4 \le \lambda_4 < \bar{\lambda}_5 \ldots,$

Ib $\lambda_1 < \bar{\lambda}_1 < \lambda_2 \le \bar{\lambda}_2 \le \lambda_3 < \bar{\lambda}_3 < \lambda_4 \le \bar{\lambda}_4 \le \lambda_5 < \bar{\lambda}_5 \ldots,$

IIa $\bar{\lambda}_1 \le \lambda_1 < \bar{\lambda}_2 < \lambda_2 \le \bar{\lambda}_3 \le \lambda_3 < \bar{\lambda}_4 < \lambda_4 \le \bar{\lambda}_5 \ldots,$

IIb $\lambda_1 \le \bar{\lambda}_1 \le \lambda_2 < \bar{\lambda}_2 < \lambda_3 \le \bar{\lambda}_3 \le \lambda_4 < \bar{\lambda}_4 < \lambda_5 \le \bar{\lambda}_5 \ldots,$

Proof: We sketch the proof here, but refer the reader to [2] for detail. It is proved that $\varphi(\lambda)$ has zeros by showing it changes sign for some λ in each of the intervals $(\bar{\lambda}_i , \bar{\lambda}_{i+1})$, where the $\bar{\lambda}_i$ are the Sturmian eigenvalues. Then, since $\varphi(\lambda)$ is continuous it must vanish in each of the intervals at some λ . Knowing that c_0u_0 is the only solution of (1.1) which satisfies the Sturmian conditions, we can reduce (1.14) to

$$\varphi(\overline{\lambda}_i) = \frac{1}{M_1[u_0(b,\overline{\lambda}_i)]} [1 - M_1 [u_0(b,\overline{\lambda}_i)]]^2 \qquad (i=1,2,\ldots) \quad (1.15)$$

where $\overline{\lambda}_i$ are the values of λ furnishing a solution to Sturm's problem. Thus, we see that unless $M_1 [u_0(b,\lambda)] = 1$ (whereby φ vanishes), φ has the same sign as $M_1 [u_0(b,\overline{\lambda}_i)]$ at $\lambda = \overline{\lambda}_1 , \overline{\lambda}_2 , \ldots$

Furthermore, we know by Sturm that $M_0 [u_0(b,\overline{\lambda})] = 0$ and $M_1 [u_0(b,\lambda)] = 0$ have alternating roots and that both change sign when they vanish. Therefore, $M_1 [u_0(b,\lambda)]$ alternates sign at $\lambda_1 , \lambda_2 , \ldots$

We thus separate the problem into two cases

$$\mathrm{I} \begin{cases} \varphi \geq 0 \text{ at } \lambda_{2n+1} \\ \varphi < 0 \text{ at } \lambda_{2n} \end{cases} \qquad n = 1,2, \ldots$$

where $M_1 [u_0 (b,\lambda_1)] > 0$

$$(1.16)$$

$$\mathrm{II} \begin{cases} \varphi < 0 \text{ at } \lambda_{2n+1} \\ \varphi \geq 0 \text{ at } \lambda_{2n} \end{cases} \qquad n = 1,2, \ldots$$

where $M_1 [u_0(b,\lambda_1)] < 0$

Now, examining (1.16) and carefully allowing for double roots (at which φ preserves its sign), we can deduce the subcases Ia , Ib , IIa , IIb and this completes the proof.

C. THE OSCILLATION THEOREM

Before proving the oscillation theorem, we must determine the sign of $M_1[u_0(b,\lambda)]$ and the sign of $\varphi(\lambda)$ for large negative λ . We deal first with $M_1[u_0(b,\lambda_1)]$. Although $M_0[u_0(b,\overline{\lambda}_1)] = 0$, this does not hold if we replace λ_1 by $\overline{\lambda} < \lambda_1$. Also, by theorems of Sturm we know

that $u_0(x,\bar{\lambda})$ does not vanish for $a < x < b$. We again have four subcases which depend on α_{11} and α_{13} .

Case I $\qquad \alpha_{11} \neq 0 \ , \ \alpha_{13} \neq 0$

Since $\alpha_{13}u_0'(b,\lambda_1) + \alpha_{14}u_0(b,\lambda_1) = 0$, (1.5) implies

$$M_1[u_0(b,\lambda_1)] = \alpha_{23} \left(- \frac{\alpha_{14}}{\alpha_{13}} u_0(b,\lambda_1)\right) + \alpha_{24}u_0(b,\lambda_1) = u_0(b,\lambda_1)/\alpha_{13}$$

However, $u_0(b,\lambda_1)$ is of the same sign as $u_0(a,\lambda_1) = \alpha_{11}$. Hence, $M_1[u_0(b,\lambda_1)]$ has the same sign as α_{11}/α_{13} . We state the results of the remaining cases as the reasoning is similar

Case II $\qquad \alpha_{11} \neq 0 \ , \ \alpha_{13} = 0$

$M_1[u_0(b,\lambda_1)]$ has the same sign as $- \alpha_{11} \alpha_{23}$.

Case III $\qquad \alpha_{11} = 0 \ , \ \alpha_{13} \neq 0$

$M_1[u_0(b,\lambda_1)]$ has the same sign as $- \alpha_{12}/\alpha_{13}$.

Case IV $\qquad \alpha_{11} = 0 \ , \ \alpha_{13} = 0$

$M_1[u_0(b,\lambda_1)]$ has the same sign as $- \alpha_{12} \alpha_{14}$.

To determine the sign of $\varphi(\lambda)$ for large negative λ , we first consider the case where $\alpha_{11} \neq 0$. Then using relations (1.7) and (1.9) of solutions of (1.1) we have

$$u_1(x,\lambda) = \left(- \frac{\alpha_{21}}{\alpha_{11}} + \int_a^x \frac{dx}{(u_0(x,\lambda))^2}\right) u_0(x,\lambda) \qquad (1.17)$$

Substituting (1.17) into (1.14) yields

$$\varphi(\lambda) = \alpha_{23}u_0'(b,\lambda) + \alpha_{24}u_0(b,\lambda)$$

$$+ \alpha_{13}\left[\left(-\frac{\alpha_{21}}{\alpha_{11}} + \int_a^b \frac{dx}{[u_0(x,\lambda)]^2}\right)u_0'(b,\lambda) + \frac{1}{u_0(b,\lambda)}\right] \tag{1.18}$$

$$+ \alpha_{14}\left[-\frac{\alpha_{21}}{\alpha_{11}} + \int_a^b \frac{dx}{[u_0(x,\lambda)]^2}\right]u_0(b,\lambda) - 2 .$$

Now, for $x > a$

$$\lim_{\lambda\to-\infty} u_0'(x,\lambda) = \lim_{\lambda\to-\infty} u_0(x,\lambda) = \infty ,$$

$$\lim_{\lambda\to-\infty} \frac{u_0'(x,\lambda)}{u_0(x,\lambda)} = \infty .$$

With these limit values of u_0 and u_0' and since $u_0'(b,\lambda)$ has the sign of $u_0(a,\lambda) = \alpha_{11}$, we have that for large negative λ the sign of $\varphi(\lambda)$ is that of $(\alpha_{11}\alpha_{23} - \alpha_{13}\alpha_{21})$, except when this term vanishes.

When $\alpha_{11} = 0$, we write $u_0(x,\lambda)$ in terms of $u_1(x,\lambda)$ and obtain the same result as above. For the case when $(\alpha_{11}\alpha_{23} - \alpha_{13}\alpha_{21}) = 0$, $\alpha_{11} \neq 0$ we use a linear combination of (1.2) such that $\alpha_{21} = \alpha_{23} = 0$ and choose a factor such that (1.5) holds to observe that for large negative λ, $\varphi(\lambda)$ has the sign of $\alpha_{11}\alpha_{13}$ provided that $\alpha_{21} = 0$, $\alpha_{22} = 1/\alpha_{11}$, $\alpha_{23} = 0$, $\alpha_{24} = 1/\alpha_{13}$. Finally, if both $(\alpha_{11}\alpha_{23} - \alpha_{13}\alpha_{21}) = 0$ and $\alpha_{11} = 0$ we have by symmetry that for large negative λ, $\varphi(\lambda)$ has the sign of $-\alpha_{21}\alpha_{23}$.

With this knowledge an oscillation theorem can be proved. Details of the proof are in [2].

Theorem 1.2:

The solution $u_p(x)$ of (1.1), (1.2) which corresponds to $\lambda = \bar{\lambda}_p$ vanishes $p-1$, p , or $p+1$ times for $a \leq x \leq b$ in accordance with the following table, in which

$$\bar{\alpha}_{11} = \alpha_{13}\,\alpha_{21} - \alpha_{11}\,\alpha_{23}\,,\ \bar{\alpha}_{12} = \alpha_{13}\,\alpha_{22} - \alpha_{12}\,\alpha_{23}\,,\ K = -\frac{\bar{\alpha}_{12}}{\bar{\alpha}_{11}}$$

$\bar{\alpha}_{11}$	p	$\frac{u_p'(a)}{u_p(a)}$	$u_p(x)$ vanishes
> 0	$2m$	$\leq K$	$p + 1$ times
> 0	$2m$	$> K$	p times
> 0	$2m + 1$	$\leq K$	p times
> 0	$2m + 1$	$> K$	$p - 1$ times
< 0	$2m$	$\leq K$	p times
< 0	$2m$	$> K$	$p - 1$ times
< 0	$2m + 1$	$\leq K$	$p + 1$ times
< 0	$2m + 1$	$> K$	p times
$\left\{\begin{matrix} = 0 \\ \bar{\alpha}_{12} > 0 \end{matrix}\right\}$	$2m$		p times
	$2m + 1$		$p + 1$ times
$\left\{\begin{matrix} = 0 \\ \bar{\alpha}_{12} < 0 \end{matrix}\right\}$	$2m$		$p + 1$ times
	$2m + 1$		p times

The above theorems are based upon hypotheses which are fairly complicated and are proved via complicated methods. As the succeeding chapters show, however, when the hypotheses are modified to fit other special cases, the resulting conclusions must also be modified accordingly.

In particular, we prove that results similar to Birkhoff's hold when the boundary conditions are altered by the introduction of an integral term; and when the problem is not self-adjoint. Also, similar conclusions can be proved when the problem is non-linear, but we do not study this case. Thus, we show what happens when these changes are made.

III. THE THEORY OF G. J. HALTINER APPLIED TO THE PROBLEM OF BIRKHOFF

A. FORMULATION OF THE ADJOINT

Having proved the existence of non-trivial solutions for the problem (1.1), (1.2) we would now like to be able to find these solutions. To do this, we first construct the adjoint problem.

If $u(x)$ and $v(x)$ are twice differentiable functions, but otherwise arbitrary, we consider the integral

$$\int_a^b vLu\,dx \;. \tag{2.1}$$

Upon integrating by parts, whereby we clear (2.1) of derivatives of u and introduce those of v, we obtain

$$\int_a^b vLu\,dx = \int_a^b uL^*v\,dx + Q(u,v,x) \tag{2.2}$$

where $L^*(v,\lambda) = v''(x) + [p_2(x) + \lambda r(x)]\,v(x)$ and (2.3)

$$Q(u,v,x) = u'(x)v(x) - u(x)v'(x) \;. \tag{2.4}$$

Equation (2.2) is called Green's formula and can be expressed as the Lagrange identity

$$vL(u,\lambda) - uL^*(v,\lambda) = dQ(u,v,x)/dx \;. \tag{2.5}$$

Equation (2.3) defines the adjoint operator L^* of (1.1). Also, in this case since $L = L^*$, the problem is formally self-adjoint. Then (2.3) together with the equations

$$N_k = \sum_{i=1}^{2} \nu_{3-i} \alpha_{ik} \qquad (k=1,2,3,4) \tag{2.6}$$

is defined as the adjoint of the boundary system (1.1), (1.2). However, since

$$\begin{aligned} N_k &= (-1)^{k-1} v^{(k-1)}(x)\Big|_{x=a} & k &= 1,2 \\ N_k &= (-1)^{k} v^{(k-3)}(x)\Big|_{x=b} & k &= 3,4 \end{aligned} \tag{2.6a}$$

the ν_i are defined. Thus the adjoint boundary conditions are written in terms of six unknowns, the ν_i and N_k, related by the four equations (2.6a). The ν_i are independent of x, but may be functions of λ. Since $\nu_i = 0$ for all i implies $v(x) \equiv 0$, we assume $\nu_i \neq 0$ for some $i=1,2$.

B. SOLUTION OF THE SYSTEM AND ITS ADJOINT

At this point, one might conclude that all we have accomplished is to create another system which is equally difficult to solve. However, it becomes clear that as a result of the special relationship, "adjointness," between the two problems, we can proceed to a solution of both problems.

Now suppose $\varphi_1(x,\lambda)$, $\varphi_2(x,\lambda)$ form any complete set of solutions of (1.1) which are analytic in λ. Then the general solution of (1.1) may be written as

$$u(x) = h_1 \varphi_1(x,\lambda) + h_2 \varphi_2(x,\lambda) . \tag{2.7}$$

We now look for solutions $\Psi_j(x)$ of the adjoint equation which have certain very nice properties. Let x_{ij} be the value of the $(i-1)$st derivative of the desired Ψ_j at the point $x = a$. Then if

$$\begin{aligned} J(\varphi_1, x_{1j}, x_{2j}, a) &\equiv \varphi_1'(a)x_{1j} - \varphi_1(a)x_{2j} \\ J(\varphi_2, x_{1j}, x_{2j}, a) &\equiv \varphi_2'(a)x_{1j} - \varphi_2(a)x_{2j} \end{aligned} \tag{2.8}$$

we look for the solution of the system of equations

$$J(\varphi_i , x_{1j} , x_{2j} , a) = \delta_{ij} . \tag{2.9}$$

Since for each j, with (2.8) considered as a linear system for the x_{ij}, the determinant of the coefficients is the Wronskian of the φ_i, we know (2.9) has a unique solution for the initial values x_{ij}. But

$$J(\varphi_i , \Psi_j, \Psi_j' , a) \equiv Q(\varphi_i(a) , \Psi_j(a))$$

by (2.4) and since φ and Ψ satisfy homogeneous equations, we have by (2.5)

$$\frac{dQ}{dx} = 0 .$$

So, (2.9) implies that

$$Q(\varphi_i , \Psi_j , x) = \delta_{ij} \tag{2.10}$$

for all x.

Considering (2.10) as a linear system for the Ψ_j^{ℓ}, we have the solution

$$\Psi_1(x) = \frac{-\varphi_2(x,\lambda)}{W(\varphi)}$$
$$\Psi_2(x) = \frac{\varphi_1(x,\lambda)}{W(\varphi)} \tag{2.10a}$$

where $W(\varphi)$ is the Wronskian of the φ_j , $j=1,2$. The set $\Psi_1(x)$, $\Psi_2(x)$ is a pair of solutions to the adjoint equations with the property that their values at $x = a$ satisfy (2.9).

Now we need to prove the linear independence of the Ψ_i in order to insure that they form a complete set of solutions. If this were not so, there would exist constants c_i , non-zero, such that $c_1 \Psi_1 + c_2 \Psi_2 \equiv 0$, for all x . In addition, $c_1 \Psi_1' + c_2 \Psi_2' = 0$ for all x . However, using relation (2.9), we see the $c_i = 0$, $i = 1,2$. Thus we can write the general solution of (2.3) in the form

$$v(x,\lambda) = d_1 \Psi_1(x,\lambda) + d_2 \Psi_2(x,\lambda) \tag{2.11}$$

where d_1 , d_2 may be functions of λ but are constant with respect to x . Then using (2.10) and (2.11) we have

$$d_j = Q(\varphi_j, v, x) \qquad j = 1,2 \tag{2.12}$$

We now prove that there are four equations for the d_j and ν_j , instead of the N_k and ν_i , with coefficients in terms of the φ_i , not the Ψ_i . Multiply the first two boundary relations of (2.6a) by $\varphi_j^{(2-k)}(a)$, $k = 1,2$, (for fixed j) and add them. Then, the right side is $Q(\varphi_j,v,a)$ or by (2.12), δ_{ij} . Repeating this process for the other value of j , we have the set of equations

$$d_j = \sum_{k=1}^{2} \sum_{i=1}^{2} \nu_{3-i} \, \alpha_{ik} \, \varphi_j^{(2-k)}(a) \qquad j = 1,2 . \tag{2.13}$$

If we define

$$A^a_{ij}(\lambda) = \sum_{k=1}^{2} \alpha_{ik}\, \varphi_j^{(2-k)}(a) \qquad i,j = 1,2 \tag{2.14}$$

we can write (2.13) as

$$d_j = \sum_{i=1}^{2} \nu_{3-i}\, A^a_{ij}(\lambda) \qquad j = 1,2\ . \tag{2.13a}$$

Similarly, using $-\varphi_j^{(2-k)}(b)$, $k = 3,4$ as the multiplier for the remaining two boundary relations of the system (2.6a), we have

$$d_j = -\sum_{k=3}^{4}\sum_{i=1}^{2} \nu_{3-i}\, \alpha_{ik}\, \varphi_j^{(4-k)}(b) \qquad j = 1,2\ . \tag{2.15}$$

Again, if we define

$$\overset{b}{A}_{ij}(\lambda) = \sum_{k=3}^{4} \alpha_{ik}\, \varphi_j^{(4-k)}(b) \qquad j = 1,2\ , \tag{2.16}$$

we can write (2.15) as

$$d_j = -\sum_{i=1}^{2} \nu_{3-i}\, A^b_{ij}\ (\lambda) \qquad j = 1,2 \tag{2.15a}$$

The linear combinations of the boundary conditions of the adjoint system, which resulted in the four relations given by (2.13a) and (2.15a), have determinants that are precisely $W(\varphi(a))$ and $W(\varphi(b))$. Thus the four boundary conditions given by (2.13a) and (2.15a) in terms of d_j and ν_i , together with (2.11) are equivalent to the system (2.6a).

Now consider any solution of the boundary problem (1.1), (1.2) where $A_1\ (u,\lambda)$ represents (1.2a) and $A_2(u,\lambda)$ represents (1.2b). This solution must be of the form (2.7) which when substituted in the boundary conditions yields the linear system

$$\sum_{j=1}^{2} h_j\, A_{ij}(\lambda) = 0 \qquad i = 1,2 \tag{2.17}$$

where the $A_{ij}(\lambda)$ represent the quantities $A_i(\varphi_j(x,\lambda))$. Then a non-trivial solution for the h_j exists if and only if λ is an eigenvalue for $\Delta(\lambda) = 0$, where $\Delta(\lambda)$ is the determinant of the $A_{ij}(\lambda)$.

We know that the number of eigenvalues associated with $\Delta(\lambda)$ in any finite region is finite since $\Delta(\lambda)$ is an analytic function of λ. Therefore, the eigenvalues may be arranged in a sequence in order of increasing absolute magnitude,

$$\lambda_1 , \lambda_2 , \lambda_3 , \ldots . \tag{2.18}$$

where $|\lambda_r| \le |\lambda_{r+1}|$ for all r. If an eigenvalue of index r exists, it appears r consecutive times in the sequence. We now show that the eigenvalues of the two adjoint boundary value problems are the same.

Let λ_m be a member of the sequence (2.18). Then by the above discussion there exists a non-trivial solution for the ν_j^m of the linear system

$$\sum_{i=1}^{2} \nu_{3-i}^m A_{ij}(\lambda_m) = 0 \qquad j = 1,2 . \tag{2.19}$$

Then we can find a unique function $v_m(x)$ satisfying the first three equations of (2.6a) with $\lambda = \lambda_m$ and $\nu_j = \nu_j^m$, $j = 1,2$. Therefore, equations (2.13a) are fulfilled and together with (2.19), the equations (2.15a) are satisfied at $\lambda = \lambda_m$ and $\nu_j = \nu_j^m$. Now, combining (2.12) and (2.19) yields

$$Q(\varphi_j , v_m , b) = - \sum_{k=3}^{4} \sum_{i=1}^{2} \nu_{3-i}^m \alpha_{ik} \varphi_j^{(4-k)}(b) \qquad j = 1,2$$

Then using (2.4) we have

$$\sum_{k=3}^{4} \varphi_j^{(4-k)}(b) \left[\sum_{i=1}^{2} \alpha_{ik} \nu_{3-i} - N_k \right] = 0 \qquad j = 1,2$$

where the N_k are as in (2.6a). Since $W(\varphi(b)) \neq 0$, we must have

$$N_k = \sum_{i=1}^{2} \alpha_{ik} \nu_{3-i}^{m} \qquad k = 3,4$$

and thus the remaining two boundary conditions of (2.6a) are satisfied. Hence, the entire adjoint system is satisfied by $v_m(x)$ at $\lambda = \lambda_m$ and $\nu_j = \nu_j^m$. Therefore, every eigenvalue of the original system is also an eigenvalue of the adjoint system.

To prove the converse, let λ be an eigenvalue of the adjoint boundary problem. Then there exist $v(x)$ and ν_1, ν_2 satisfying (2.6a). Thus, by (2.13a) and (2.15a)

$$\sum_{i=1}^{2} \nu_{3-i} A_{ij}(\lambda) = 0 \qquad j = 1,2$$

But the trivial solution is excluded so that not all the ν_i are zero. Then we must have that the determinant of the $A_{ij}(\lambda)$ is zero which implies that λ is an eigenvalue of the original system.

In summary, we have first written the adjoint equation for (1.1) in a novel form. Namely, in terms of the six unknowns, the ν_i and N_k, related by the four equations (2.6a). We then proved the ability to write the solutions of the adjoint problem in terms of the solutions to the original problem. Then, if we write

$$v(x) = \sum_{j=1}^{2} d_j \Psi_j(x)$$

there are four equations for the d_j and ν_i, instead of the N_k and ν_i, with coefficients in terms of the φ_i. Thus, the eigenvalues of the adjoint system are determined by the φ_i; and therefore, the eigenvalues, whose existence is known from our previous chapter, are the same for both solutions.

C. THE GREEN'S FUNCTION

Before determining the Green's function, it is important to understand its relationship to the given problem. Given a linear, inhomogeneous second order differential equation there exists an integral operator G,

$$G(r) = \int_a^x G(x,\xi)r(\xi)d\xi \tag{2.20}$$

such that $G(r) = u$, where $r(x)$ is the forcing function of the differential equation. In addition, $G(r) = u$ satisfies the given homogeneous boundary conditions. However, the Green's function is a non-eigenvalue problem. In fact, we have by a familiar theorem [3, Chapter 10] that if the forcing function of a second order inhomogeneous linear differential equation is continuous and if the boundary value problem does not admit an eigenfunction, then the function which is the kernel of (2.20) is a Green's function for the system (1.1), (1.2).

We make a study of this non-eigenvalue problem here because Haltiner has obtained a novel representation of the Green's function. First, it is dependent on the eigenfunctions determined in the preceding parts of this Chapter. Second, Haltiner has effectively separated the Green's function into two component functions.

The conventional approach to determining this function is to solve the adjoint equation having a forcing term which is a "Dirac delta function," utilizing the continuity of the function and the fact that the derivative has a jump condition. For the self-adjoint problem we would expect the Green's function to have the form

$$G(x,\xi,\lambda) = \begin{cases} U(x,\lambda)V(\xi,\lambda)/W(\xi,\lambda) & a \le x \le \xi \\ U(\xi,\lambda)V(x,\lambda)/W(\xi,\lambda) & \xi \le x \le b \end{cases}$$

where W is the Wronskian of U and V .

However, the representation due to Haltiner has the form

$$G(x,\xi,\lambda) = g_1(x,\xi,\lambda) + g_2(x,\xi,\lambda)/\Delta(\lambda) \tag{2.21}$$

$$g_1(x,\xi,\lambda) = \begin{cases} -\sum_{i=1}^{2} \varphi_i(x,\lambda)\,\Psi_i(\xi,\lambda) & a \le x \le \xi \\ 0 & \xi \le x \le b \end{cases} \tag{2.21a}$$

$$g_2(x,\xi,\lambda) = -1 \begin{vmatrix} \varphi_j(x,\lambda) & 0 \\ A_{ij}(\lambda) & \sum_{k=1}^{2} A^a_{ik}\,\Psi_k(\xi) \end{vmatrix} \quad i,j = 1,2 \tag{2.21b}$$

where the elements of the determinant for g_2 are as previously defined. The significance of this representation is that g_1 turns out to be the Green's function for an initial value problem. Surprisingly, the difference between G and g_1 is precisely $g_2/\Delta(\lambda)$.

It is by no means obvious that these two representations are equivalent, so we consider an example:

$$u'' + \lambda u = 0 \qquad 0 \le x \le 1 \tag{2.22a}$$

$$u'(0) + u(0) + u(1) = 0 \tag{2.22b}$$

$$2u'(0) + u(0) + u(1) = 0 \tag{2.22c}$$

Then, by the conventional method we find the Green's function by solving the boundary value problem

$$g'' + \lambda g = \delta(x-\xi) \qquad 0 \le x \le 1 \tag{2.23}$$

$$g'(0) + g(0) + g(1) = 0 \tag{2.24a}$$

$$2g'(0) + g(0) + g(1) = 0 \tag{2.24b}$$

We know that any solution to this boundary value problem is of the form

$$g(x,\lambda) = \begin{cases} A \sin \sqrt{\lambda}x + B \cos \sqrt{\lambda}x & 0 \le x \le \xi \\ C \sin \sqrt{\lambda}x + D \cos \sqrt{\lambda}x & \xi \le x \le 1 \end{cases} \tag{2.25}$$

Where the coefficients A,B,C,D are to be determined using the boundary conditions. Using equation (2.24), we find that $A = 0$. We now make use of the fact that $g(x,\lambda)$ satisfies two conditions at $x = \xi$. First, $g(x,\lambda)$ is continuous at $x = \xi$. Second, $dg(x,\lambda)/dx$ has a jump condition with value 1 at $x = \xi$. Thus we have the three equations

$$B \cos \sqrt{\lambda}\xi - C \sin \sqrt{\lambda}\xi - D \cos \sqrt{\lambda}\xi = 0$$

$$B \sqrt{\lambda} \sin \sqrt{\lambda}\xi + C \sqrt{\lambda} \cos \sqrt{\lambda}\xi - D \sqrt{\lambda} \sin \sqrt{\lambda}\xi = 1$$

$$B + C \sin \sqrt{\lambda} + D \operatorname{cox} \sqrt{\lambda} = 0$$

Applying Cramer's rule we can solve for the coefficients

$$B = \frac{\cos \sqrt{\lambda} \sin \sqrt{\lambda}\xi - \sin \sqrt{\lambda} \cos \sqrt{\lambda}\xi}{\sqrt{\lambda}\,(1 + \cos \sqrt{\lambda})}$$

$$C = \frac{\cos \sqrt{\lambda}\xi}{\sqrt{\lambda}}$$

$$D = \frac{-\,(\sin \sqrt{\lambda} \cos \sqrt{\lambda}\xi + \sin \sqrt{\lambda}\xi)}{\sqrt{\lambda}\,(1 + \cos \sqrt{\lambda})}$$

Therefore, substituting these values into (2.25) we have the Green's function

$$G(x,\xi,\lambda) = \begin{cases} \dfrac{\cos \sqrt{\lambda} \sin \sqrt{\lambda}\xi - \sin \sqrt{\lambda} \cos \sqrt{\lambda}\xi}{\sqrt{\lambda}\,(1 + \cos \sqrt{\lambda})} \cos \sqrt{\lambda}x & 0 \le x \le \xi \\[2ex] \dfrac{\cos\sqrt{\lambda}\xi \sin\sqrt{\lambda}x}{\sqrt{\lambda}} - \dfrac{(\sin\sqrt{\lambda}\cos\sqrt{\lambda}\xi + \sin\sqrt{\lambda}\xi)}{\sqrt{\lambda}\,(1 + \cos \sqrt{\lambda})} \cos\sqrt{\lambda}x & \xi \le x \le 1 \end{cases} \tag{2.26}$$

Now we examine Haltiner's definition. A complete set of solutions for (2.22a) is as stated above,

$$\varphi_1(x,\lambda) = \sin\sqrt{\lambda}x \qquad \varphi_2(x,\lambda) = \cos\sqrt{\lambda}x$$

Then using equations (2.10a) we have

$$\Psi_1(x,\lambda) = \cos\sqrt{\lambda}x/\sqrt{\lambda} \quad , \quad \Psi_2(x,\lambda) = -\sin\sqrt{\lambda}x/\sqrt{\lambda}$$

Substitution of these functions into (2.21a) yields

$$g_1(x,\xi,\lambda) = \begin{cases} (\cos\sqrt{\lambda}x \sin\sqrt{\lambda}\xi - \sin\sqrt{\lambda}x \cos\sqrt{\lambda}\xi)/\sqrt{\lambda} & 0 \le x \le \xi \\ 0 & \xi \le x \le 1 \end{cases} \tag{2.27}$$

We now find the elements of the determinant in (2.21b). Recalling the definition of $A_{ij}(\lambda)$ from (2.17) we have

$$\begin{aligned} A_{11}(\lambda) &= \sqrt{\lambda} + \sin\sqrt{\lambda} \qquad & A_{12}(\lambda) &= 1 + \cos\sqrt{\lambda} \\ A_{21}(\lambda) &= 2\sqrt{\lambda} + \sin\sqrt{\lambda} \qquad & A_{22}(\lambda) &= 1 + \cos\sqrt{\lambda} \end{aligned} \tag{2.28}$$

Also by (2.14)

$$A_{11}^0 = \sqrt{\lambda} \qquad A_{12}^0 = 1$$

$$A_{21}^0 = 2\sqrt{\lambda} \qquad A_{22}^0 = 1$$

Then we know

$$\sum_{k=1}^{2} A_{1k}^0 \Psi_k(\xi) = \cos\sqrt{\lambda}\xi - \frac{\sin\sqrt{\lambda}\xi}{\sqrt{\lambda}}$$

$$\sum_{k=1}^{2} A_{2k}^0 \Psi_k(\xi) = 2\cos\sqrt{\lambda}\xi - \frac{\sin\sqrt{\lambda}\xi}{\sqrt{\lambda}}$$

Thus,

$$g_2(x,\xi,\lambda) = - \begin{vmatrix} \sin\sqrt{\lambda}x & \cos\sqrt{\lambda}x & 0 \\ \sqrt{\lambda} + \sin\sqrt{\lambda} & 1 + \cos\sqrt{\lambda} & \cos\sqrt{\lambda}\xi - \dfrac{\sin\sqrt{\lambda}\xi}{\sqrt{\lambda}} \\ 2\sqrt{\lambda} + \sin\sqrt{\lambda} & 1 + \cos\sqrt{\lambda} & 2\cos\sqrt{\lambda}\xi - \dfrac{\sin\sqrt{\lambda}\xi}{\sqrt{\lambda}} \end{vmatrix}$$

which when evaluated yields

$$g_2(x,\xi,\lambda) = \cos\sqrt{\lambda}x(\sin\sqrt{\lambda}\xi+\sin\sqrt{\lambda}\cos\sqrt{\lambda}\xi) - (1+\cos\sqrt{\lambda})\sin\sqrt{\lambda}x\cos\sqrt{\lambda}\xi$$

Finally, we determine $\Delta(\lambda)$ by finding the determinant of the elements in (2.28). Thus

$$\Delta(\lambda) = -\sqrt{\lambda}\,(1 + \cos\sqrt{\lambda})$$

Therefore,

$$\frac{g_2(x,\xi,\lambda)}{\Delta(\lambda)} = \frac{\sin\sqrt{\lambda}x\cos\sqrt{\lambda}\xi}{\sqrt{\lambda}} - \frac{\cos\sqrt{\lambda}x\,(\sin\sqrt{\lambda}\xi + \sin\sqrt{\lambda}\cos\sqrt{\lambda}\xi)}{\sqrt{\lambda}\,(1 + \cos\sqrt{\lambda})} \tag{2.29}$$

Now by (2.21), we have after simplifying

$$G(x,\xi,\lambda) = \begin{cases} \dfrac{(\cos\sqrt{\lambda}\sin\sqrt{\lambda}\xi - \sin\sqrt{\lambda}\cos\sqrt{\lambda}\xi)\cos\sqrt{\lambda}x}{\sqrt{\lambda}\,(1 + \cos\sqrt{\lambda})} & 0 \le x \le \xi \\[2ex] \dfrac{\cos\sqrt{\lambda}\xi\sin\sqrt{\lambda}x}{\sqrt{\lambda}} - \dfrac{(\sin\sqrt{\lambda}\xi + \sin\sqrt{\lambda}\cos\sqrt{\lambda}\xi)\cos\sqrt{\lambda}x}{\sqrt{\lambda}\,(1 + \cos\sqrt{\lambda})}, & \xi \le x \le 1 \end{cases}$$

Thus, we have obtained the Green's function by using Haltiner's definition. However, it is a more time consuming method than the conventional one. Thus despite the fact that it is a very interesting way to write the Green's function, its advantage over the conventional method is not apparent.

IV. THE LINEAR PROBLEM OF W. M. WHYBURN

A. STATEMENT OF THE PROBLEM

By making the proper substitution, we can write (1.1) as an equivalent system of first order differential equations

$$\frac{dy}{dx} = K(x,\lambda)z \qquad \frac{dz}{dx} = G(x,\lambda)y \tag{3.1}$$

It is convenient to specify the domain of x as $X: a \le x \le b$, $-\infty < a < b < \infty$; and a domain for λ as $L: \lambda^* - \delta < \lambda < \lambda^* + \delta$, $0 < \delta \le \infty$. We now consider conditions where the right hand condition is an integral condition; specifically

$$\alpha(\lambda)\, z(a,\lambda) - \beta(\lambda)\, y(a,\lambda) = 0 \tag{3.2a}$$

$$J(b,\lambda) = 0 \tag{3.2b}$$

where
$$J(x,\lambda) = \int_a^x j(t,\lambda)\, y\,(t,\lambda)\, dt$$

We make note of the fact that (3.1), (3.2) can be made into a boundary value problem which has the form of (1.1), (1.2), but which violates condition (1.3). In particular, if $j(t,\lambda) = G(t,\lambda)$ we have the non-self-adjoint system (3.1), (3.3) where

$$\begin{aligned} \alpha(\lambda)\, z(a,\lambda) - \beta(\lambda)\, y(a,\lambda) &= 0 \\ z(a,\lambda) &= z(b,\lambda) \end{aligned} \tag{3.3}$$

The functions mentioned above are assumed to satisfy the following hypotheses:

H1: For each x on X, $K(x,\lambda)$, $G(x,\lambda)$, $j(x,\lambda)$ are each continuous on L.

H2: For each λ on L, $K(x,\lambda)$, $G(x,\lambda)$, $j(x,\lambda)$ are each measurable on X.

H3: There exists a Lebesgue integrable function $M(x)$ on X such that $|K(x,\lambda)| \leq M(x)$, $|G(x,\lambda)| \leq M(x)$ and $|j(x,\lambda)| \leq M(x)$ on $X \times L$.

H4: The functions $\alpha(\lambda)$ and $\beta(\lambda)$ are continuous in λ on L.

H5: $K(x,\lambda) > 0$ on $X \times L$.

H6: The coefficients K, G, α, β, satisfy conditions that are sufficient to insure the validity of existence and uniqueness theorems for the system (3.1), (3.2a) and

$$y(b,\lambda) = 0 \tag{3.4}$$

Hypotheses one through four require that the coefficients be well-behaved functions of x and λ; thus enabling the proof of the existence of a non-trivial solution pair. They are by no means unreasonable restrictions. Perhaps the most restrictive is hypothesis five. There are many problems which can easily fulfill all the other hypotheses except this one. Hypothesis six is vital because we use the zeros of (3.1), (3.2a), (3.4) to isolate the zeros of our problem.

B. THE EXISTENCE THEOREM

To prove an existence theorem for (3.1), (3.2) we examine the simpler problem (3.1), (3.2a), (3.4). We then have three cases to study. Case I: $\alpha(\lambda) \neq 0$ on L. By an existence theorem due to Caratheodory [4], we are assured of having a function pair $(y(x,\lambda), z(x,\lambda))$ which satisfies (3.1) and

$$y(x,\lambda) = \alpha(\lambda) \qquad z(x,\lambda) = \beta(\lambda)$$

Using this pair of functions, we again use the known zeros of $y(x)$ to isolate the zeros of $J(x)$. First, let $x_1 < x_2 < \ldots < x_n$ be the zeros of $y(x)$ on X -- since all the zeros are simple and there are only a finite number. The integral $J(x)$ is then divided into a sum of integrals:

$$I_i = \int_{x_i}^{x_{i+1}} jy \, dt \qquad\qquad i = 0,1, \ldots, n$$

where $x_0 = a$ and $x_{n+1} = b$ and y is the solution of (3.1), (3.2a), (3.4). Then for $t = 0,1, \ldots, n-1$ we have $I_t I_{t+1} < 0$. If G/K is negative and a non-increasing function of x on X, we can prove that $|I_0| < |I_1| \le \ldots \le |I_{n-1}|$. This implies that $J(x_i) \cdot J(x_{i+1}) < 0$ and hence that $J(x)$ has one and only one zero on the interval $x_i \le x \le x_{i+1}$. We then have the following separation theorem due to Whyburn [17].

Theorem 4.1:

If for a fixed value of λ on L, G/K is negative and a non-increasing function of x on X, and j/G is positive and a non-decreasing function of x on X, then the zeros of $J(x)$ and the zeros of $y(x)$ separate each other on X, where $y(x)$ is the solution of (3.1), (3.2a), (3.4).

Other authors [7] have proven that the zeros of $J(x,\lambda)$ are continuous functions of λ on L. For any fixed λ that is greater than or equal to λ_0 (where λ_0 is the first eigenvalue of the system (3.1), (3.2a), (3.4)), we know that the zeros of $J(x,\lambda)$ and $y(x,\lambda)$ separate each other on X. Suppose $x_i(\lambda)$ is the i^{th} zero of $J(x,\lambda)$ on (a,b') where $b' > b$. The definition of the coefficients of the system is that they have the same values outside of (a,b) that they have at $x = b$. Recall that we want $J(b,\lambda) = 0$. Then since $x_i(\lambda)$ is

continuous on L, we show that it varies continuously from some value less than b to some value greater than b. Thus it is necessary to extend the interval. If we let λ increase from λ_i to λ_{i+1}, $x_i(\lambda)$ varies from $x_i(\lambda_i) > b$ to $x_i(\lambda_{i+1}) < b$. Hence, there must be at least one value of λ, $\lambda_i < \lambda < \lambda_{i+1}$, such that $x_i(\lambda) = b$. Let k_i be the set of values of λ for which $x_i(\lambda) = b$. Then $\lambda_i < k_i < \lambda_{i+1}$. Note that k_i may be an aggregate of a finite or infinite number of values. We, therefore, can state

Theorem 4.2:

If $\lambda_0 < \lambda_1 < \lambda_2$, ... are the ordered eigenvalues of (3.1), (3.2a), (3.4) and if the hypotheses of Theorem 4.1 hold for every λ on $\lambda_0 \leq \lambda < \lambda^* + \delta$, then there exists an infinite set of eigenvalues, k_0, k_1, k_2, ..., for the system (3.1), (3.2) such that $\lambda^* - \delta < \lambda_0 < k_0 < \lambda_1 < k_1 < \ldots < \lambda^* + \delta$.

C. THE OSCILLATION THEOREM

With these two theorems we can now easily prove the oscillation theorem due to Whyburn.

Theorem 4.3:

If p_i is an eigenvalue of (3.1), (3.2), then, on $a \leq x \leq b$, $J(x,p_i)$ has exactly i zeros, $y(x,p_i)$ has exactly $i+1$ zeros, and $z(x,p_i)$ has either i, $i+1$, or $i+2$ zeros.

Proof: As an example, suppose we examine the zeros of $\cos \lambda x$ on the interval $(0,\pi)$. As the value of λ changes, the position of the zeros on the interval changes. However, since at $x = 0$ the function does not vanish we will not lose any zeros on the left as the value of λ changes. Similarly, in our case since $y(a,\lambda) \neq 0$, no zeros of y or of J are lost from X on the left. As one and only one zero of y

enters the X interval every time λ passes an eigenvalue of (3.1), (3.2a), (3.4) a simple counting shows the theorem holds for $y(x,p_i)$. Then, by the separation theorem, the theorem holds for $J(x,p_i)$, $z(x,p_i)$.

The above discussion holds only for Case I. We now study the other cases. Case II: $\alpha(\lambda) \equiv 0$ and for all λ on L there exists a neighborhood of $x = a$ that is a subset of X , throughout which one of the quantities j/G , K/G actually increases as x increases.

This case is handled by the same method as for Case I.

Case III: $\alpha(\lambda) \equiv 0$ and j/G and K/G are constant throughout X for every fixed λ on L .

When this occurs the zeros of $J(x,\lambda)$ and the zeros of $y(x,\lambda)$ coincide. Then if we have λ_0 , λ_1 , λ_2 , ... as the eigenvalues of (3.1), (3.2a), (3.4), the eigenvalues of (3.1), (3.2) are λ_1 , λ_3 , λ_5 , ... Also, $y(x,\lambda_i)$ has exactly i zeros on (a,b) and $F(x,\lambda_i)$ has $(i-1)/2$ zeros on (a,b).

Thus, we have existence and oscillation theorems for the second order linear differential boundary value problem of the form (3.1), (3.2). These theorems extend Birkhoff's work in that we can now consider certain non-self-adjoint problems. However, we are restricted by the condition that $K(x,\lambda) > 0$ on $X \times L$. This, though, is better than $K(x,\lambda) \equiv 1$ as with Birkhoff. By comparison, the derived standard non-self-adjoint boundary value problem (3.1), (3.3) must have $\alpha_{22} \equiv \alpha_{24} \equiv 0$, which is less general than Birkhoff's boundary conditions for the self-adjoint problem.

Thus, Whyburn's theorems are in some ways superior to Birkhoff's. However, there are still problems to which they are not applicable. In particular, we cannot handle the equation

$$u'' + \lambda u = 0$$

with boundary conditions of the form (3.3) with Whyburn's theorem, although we can via Birkhoff's theorems.

V. THE THEORY OF ETGEN AND TEFTELLER AND ITS GENERALIZATION

A. PRELIMINARIES TO THE THEOREM

We now consider a more general type of boundary conditions. They are an extension of those of Whyburn [17] in that the second condition depends not only on an integral term, but also on the value of the solution function and its derivative at both endpoints.

As in the previous chapter a system of first order equations is studied. The equations are of the form

$$\frac{dy}{dx} = k(x,\lambda)z \qquad \frac{dz}{dx} = g(x,\lambda)y \tag{4.1}$$

where the coefficient functions are real-valued on $X \times L$ where $X = (x : a \le x \le b)$ and $L = (\lambda : \lambda^* - \delta < \lambda < \lambda^* + \delta)$, $-\infty < a < b < \infty$, $0 < \delta < \infty$, with the boundary conditions

$$\alpha(\lambda)\, y(a,\lambda) - \beta(\lambda)\, z(a,\lambda) = 0 \tag{4.2a}$$

$$\gamma_1(\lambda)\, y(a,\lambda) + \delta_1(\lambda)\, z(a,\lambda) = \gamma_2(\lambda)\, y(b,\lambda) + \delta_2(\lambda)\, z(b,\lambda) + H(b,\lambda) \tag{4.2b}$$

where $H(x,\lambda) = \int_a^x h(t,\lambda)\, z(t,\lambda)\, dt$.

The hypotheses H1 - H6 of Chapter 4 are again assumed with the following additions:

1. H4 is extended to include the $\gamma_i(\lambda)$, $\delta_i(\lambda)$, $i = 1,2$.

2. Without loss of generality, we normalize condition (4.2a) by taking

$$\alpha^2(\lambda) + \beta^2(\lambda) \equiv 1 \quad \text{on} \quad L$$

The existence theorem of Etgen and Tefteller [6] requires, in addition to the previously stated six hypotheses, that the quotient h/k satisfies conditions allowing the application of a mean value theorem for integrals. Also, the sum of the squares of the coefficients in the left hand side of (4.2b) is a positive valued function no greater than 1; and that the coefficient of $y(b,\lambda)$ is greater than or equal to one. These requirements are made in order to establish crucial bounds on the function to be zeroed. Finally, the function $r(x,\lambda)$ used in the Prüfer substitution (4.4) must have the properties

$$r(b,\lambda) \geq 1 \text{ on } L$$

$$r(b,\lambda) \geq r(x,\lambda) \text{ on } X \quad \text{for each} \quad \lambda \quad \text{on} \quad L\,.$$

Once the problem has been transferred to the trigonometric domain, (4.2b) is shown to vanish by appealing to the bounds that are established and the continuity of the function. Then, it is easily seen that there exist non-empty sets which contain eigenvalues.

It is then shown that if the right hand bound on the angle function evaluated at $x = b$ is arbitrarily large, then there exist infinitely many of these eigenvalue-containing sets and an infinite number of eigenvalues.

To solve the problem of Whyburn studied in Chapter IV, they then proceed to prove another theorem using the same method as above. However, we shall show that the first theorem of Etgen and Tefteller can be generalized in such a way that Whyburn's boundary value problem is a special case of our improved theorem. Hence, the second theorem of Etgen and Tefteller (for Whyburn's problem) is unnecessary.

B. A GENERALIZATION OF THE RESULTS OF ETGEN AND TEFTELLER

Using H1 - H3, fundamental existence and uniqueness theorems [4, Chapter 2] can be applied to obtain the existence of a unique solution pair $(y(x,\lambda), z(x,\lambda))$ for (4.1) on $X \times L$ such that

$$y(a,\lambda) \equiv \beta(\lambda) \qquad z(a,\lambda) \equiv \alpha(\lambda) \quad \text{on} \quad L . \tag{4.3}$$

Of themselves the functions y and z have no special properties other than being solutions. Therefore, we will express them in terms of functions whose properties are well known to us. A convenient method for doing this is the Prüfer substitution (4.4).

Applying a polar coordinate transformation to the solution pair (4.1), (4.3) we define $r(x,\lambda)$ and $v(x,\lambda)$ by

$$\begin{aligned} y(x,\lambda) &= r(x,\lambda) \sin v(x,\lambda) \\ z(x,\lambda) &= r(x,\lambda) \cos v(x,\lambda) \end{aligned} \tag{4.4}$$

where $r(x,\lambda)$, $v(x,\lambda)$ are the solution of

$$\begin{aligned} dv/dx &= k(x,\lambda) \cos^2 v(x,\lambda) - g(x,\lambda) \sin^2 v(x,\lambda) \\ dr/dx &= r(x,\lambda) [k(x,\lambda) + g(x,\lambda)] \sin v(x,\lambda) \cos v(x,\lambda) \end{aligned} \tag{4.5}$$

Applying (4.3) to (4.4) and using H6

$$r(a,\lambda) = [y^2(a,\lambda) + z^2(a,\lambda)]^{\frac{1}{2}} = 1 \tag{4.6a}$$

$$\sin v(a,\lambda) = y(a,\lambda) = \beta(\lambda) \tag{4.6b}$$

$$\cos v(a,\lambda) = z(a,\lambda) = \alpha(\lambda) . \tag{4.6c}$$

We can now prove that the solution pair $(y(x,\lambda), z(x,\lambda))$ is non-trivial. First, for each λ on L, $y^2(x,\lambda) + z^2(x,\lambda) = r^2(x,\lambda)$ on X. Also, $r(x,\lambda)$ is a solution of a first order differential equation which

has a positive value at the left endpoint. Thus, we conclude $y^2(x,\lambda) + z^2(x,\lambda) > 0$ on $X \times L$. Therefore, the solution pair cannot be the trivial pair.

We now state and prove a theorem which generalizes the work of Etgen and Tefteller.

Theorem 4.1:

Let $(y(x,\lambda), z(x,\lambda))$ be the non-trivial solution of (4.1), (4.3) and let $v(x,\lambda)$ and $r(x,\lambda)$ be defined by (4.5) and (4.6). Then $v(b,\lambda) \geq 0$ on L. In addition to H1-H6, let the following conditions hold:

(i) $h(x,\lambda)/k(x,\lambda)$ is integrable, non-negative and non-decreasing on X for all λ on L.

(ii) $\gamma_2(\lambda) \geq [\gamma_1^2(\lambda) + \delta_1^2(\lambda)]^{\frac{1}{2}}$ on L.

(iii) $r(b,\lambda) \geq 1$ on L and $r(b,\lambda) \geq r(x,\lambda)$ on X for all λ on L.

Then, if m is the least non-negative integer such that $\inf_{\lambda \epsilon L} v(b,\lambda) < (2m + 1)\pi/2$, and n is an integer such that $\sup_{\lambda \epsilon L} v(b,\lambda) > (2n + 1)\pi/2$, and if $n \geq m + 1$, there exists at least p, $p = n - m$, non-empty sets of eigenvalues $T_0, T_1, \ldots, T_{p-1}$ for the boundary problem (4.1), (4.2). Moreover, the number of distinct eigenvalues for (4.1), (4.2) is at least $p/2$ if p is even and $(p + 1)/2$ if p is odd.

Proof: The assumed continuity conditions on the coefficients of the boundary problem imply $v(x,\lambda)$ is continuous on XL. Now, for any fixed λ on L, since $v(a,\lambda) \geq 0$ and whenever $y(x,\lambda) = 0$, $v'(x,\lambda) > 0$, we conclude $v(x,\lambda) \geq 0$ on X. In particular, $v(b,\lambda) \geq 0$ and it follows that $v(b,\lambda) \geq 0$ on L.

Let m and n be the integers with the properties described in the hypothesis. By the continuity of $v(b,\lambda)$, there exists a value of λ, say λ_0, such that $v(b,\lambda_0) = (2m+1)\pi/2$ and a value of λ, say λ_p, such that $v(b,\lambda_p) = (2n+1)\pi/2$. Since $\lambda_0 \neq \lambda_p$, we can assume without loss of generality that $\lambda_0 < \lambda_p$.

Substituting (4.4) into (4.2b) we obtain:

$$\gamma_1(\lambda) \sin v(a,\lambda) + \delta_1(\lambda) \cos v(a,\lambda) = r(b,\lambda)\, f(b,\lambda) + H(b,\lambda) \qquad (4.7)$$

where $$f(b,\lambda) = \gamma_2(\lambda) \sin v(b,\lambda) + \delta_2(\lambda) \cos v(b,\lambda) \qquad (4.8)$$

The left side of (4.7) is equal to

$$[\gamma_1^2(\lambda)+\delta_1^2(\lambda)]^{\frac{1}{2}} \left[\sin v(a,\lambda) \frac{\gamma_1(\lambda)}{[\gamma_1^2(\lambda)+\delta_1^2(\lambda)]^{\frac{1}{2}}} + \cos v(a,\lambda) \frac{\delta_1(\lambda)}{[\gamma_1^2(\lambda)+\delta_1^2(\lambda)]^{\frac{1}{2}}} \right] \qquad (4.9)$$

If we define $\theta(\lambda)$ such that

$$\sin \theta(\lambda) = \frac{\delta_1(\lambda)}{[\gamma_1^2(\lambda)+\delta_1^2(\lambda)]^{\frac{1}{2}}} \qquad \cos \theta(\lambda) = \frac{\gamma_1(\lambda)}{[\gamma_1^2(\lambda)+\delta_1^2(\lambda)]^{\frac{1}{2}}}$$

then (4.9) becomes

$$[\gamma_1^2(\lambda) + \delta_1^2(\lambda)]^{\frac{1}{2}} \sin [v(a,\lambda) + \theta(\lambda)] \,. \qquad (4.10)$$

Therefore, equation (4.2b) becomes

$$[\gamma_1^2(\lambda) + \delta_1^2(\lambda)]^{\frac{1}{2}} \sin [v(a,\lambda) + \theta(\lambda)] = r(b,\lambda)\, f(b,\lambda) + H(b,\lambda) \qquad (4.11)$$

Now, define $Q(\lambda)$ by

$$Q(\lambda) = r(b,\lambda)\, f(b,\lambda) + H(b,\lambda) \qquad (4.12)$$

Next, we fix λ on L and consider

$$H(b,\lambda) = \int_a^b h(t,\lambda)\, z(t,\lambda)\, dt = \int_a^b \frac{h(t,\lambda)}{K(t,\lambda)} y'(t,\lambda)\, dt$$

Condition (i) allows the application of a mean value theorem for integrals to obtain

$$\frac{h(b,\lambda)}{K(b,\lambda)} \min_{x \in X} \int_x^b y'(t,\lambda)dt \le H(b,\lambda) \le \frac{h(b,\lambda)}{K(b,\lambda)} \max_{x \in X} \int_x^b y'(t,\lambda)dt \qquad (4.13)$$

we let $\bar{x}$ and x^* be the values of x on X such that

$$\min_{x \in X} \int_x^b y'(t,\lambda)dt = \int_{\bar{x}}^b y'(t,\lambda)dt$$

$$\max_{x \in X} \int_x^b y'(t,\lambda)dt = \int_{x^*}^b y'(t,\lambda)\, dt$$

We now have

$$r(b,\lambda)\left\{f(b,\lambda) + \frac{h(b,\lambda)}{K(b,\lambda)}\left[\sin v(b,\lambda) - \frac{r(\bar{x},\lambda)}{r(b,\lambda)} \sin v(\bar{x},\lambda)\right]\right\} \le Q(\lambda)$$
$$\le r(b,\lambda)\left\{f(b,\lambda) + \frac{h(b,\lambda)}{K(b,\lambda)}\left[\sin v(b,\lambda) - \frac{r(x^*,\lambda)}{r(b,\lambda)} \sin v(x^*,\lambda)\right]\right\} \qquad (4.14)$$

but since $n = m + p$, $p \ge 1$, and since $v(b,\lambda)$ is continuous in λ , there exist $p - 1$ values of λ , $\lambda_1 , \ldots , \lambda_{p-1}$ on (λ_0 , λ_p) such that $v(b,\lambda_j) = [2(m + j) + 1]\pi/2$, $j = 1,2 , \ldots , p - 1$. Moreover, we may assume $\lambda_1 < \lambda_2 < \ldots < \lambda_{p-1}$. Now, choose any integer j , $0 \le j \le p - 1$, and without loss of generality, assume $\sin v(b,\lambda_j) = 1$. Then $\sin v(b,\lambda_{j+1}) = -1$. Then from (4.14) we have

$$Q(\lambda_j) \ge r(b,\lambda_j)\left\{\gamma_2(\lambda_j) + \frac{h(b,\lambda_j)}{K(b,\lambda_j)} [1 - \frac{r(\bar{x},\lambda_j)}{r(b,\lambda_j)} \sin v(\bar{x},\lambda_j)]\right.$$

$$\ge [\gamma_1^2(\lambda_j) + \delta_1^2(\lambda_j)]^{\frac{1}{2}} \ge [\gamma_1^2(\lambda_j) + \delta_1^2(\lambda_j)]^{\frac{1}{2}} \sin [v(a,\lambda_j) + \theta(\lambda_j)]$$

Similiarly,

$$Q(\lambda_{j+1}) \le [\gamma_1^2(\lambda_{j+1}) + \delta_1^2(\lambda_{j+1})]^{\frac{1}{2}} \sin [v(a,\lambda_{j+1}) + \theta(\lambda_{j+1})]$$

using conditions (i), (ii), (iii). Therefore, as λ increases from λ_j to λ_{j+1}, $Q(\lambda)$ changes continuously in value from not less than $[\gamma_1^2(\lambda) + \delta_1^2(\lambda)]^{\frac{1}{2}}$ to not more than the negative of the same value for $j = 0,1, \ldots , p - 1$. However, we also know that

$$[\gamma_1^2(\lambda) + \delta_1^2(\lambda)]^{\frac{1}{2}} |\sin [v(a,\lambda) + \theta(\lambda)]| \le [\gamma_1^2(\lambda) + \delta_1^2(\lambda)]^{\frac{1}{2}}$$

Thus there must be at least one value of λ on $[\lambda_j , \lambda_{j+1}]$ with the property that (4.11) is satisfied. Let $T_j = (\lambda \epsilon [\lambda_j , \lambda_{j+1}]$ such that (4.11) is satisfied) for $j = 0,1 , \ldots , p - 1$. It could happen that the λ which satisfy (4.11) are alternate endpoints. Thus there will be at least $p/2$ or $(p + 1)/2$ eigenvalues for (4.1), (4.2). This completes the proof of the theorem.

We note that a similar theorem could be proved if the function $H(x,\lambda)$ is replaced by

$$J(x,\lambda) = \int_a^x j(t,\lambda)\, y(t,\lambda)\, dt$$

Also, since in the hypotheses of the theorem, the integer n can be chosen arbitrarily large, then there exist infinitely many non-empty sets of eigenvalues for the problem (4.1), (4.2).

C. APPLICATION OF THE IMPROVED THEOREM

The foregoing theorem differs from that of Etgen and Tefteller in that condition (ii) has been relaxed. Condition (ii) of Etgen and Tefteller's theorem is, in fact, a special case of our condition (ii). Hence, their problem is a special case of our theorem. The real

significance of our new theorem is that Whyburn's problem also becomes a special case; and thus we no longer need two separate theorems.

D. A PARTICULAR PROBLEM

Suppose that the functions K and G in (4.1) are defined to be λ and $-\lambda$, respectively. Also, let the X interval be $[0,\pi]$ and the L interval be $[\pi, 2\pi]$ with the boundary conditions

$$\tfrac{1}{2}y(0,\lambda) - \sqrt{3}/2z(0,\lambda) = 0 \tag{4.15a}$$

$$\sqrt{\tfrac{1}{2}}\, y(0,\lambda) + \sqrt{\tfrac{1}{2}}\, z(0,\lambda) = y(\pi,\lambda) + z(\pi,\lambda) + H(\pi,\lambda) \tag{4.15b}$$

In addition, $h(t,\lambda) = \lambda$, which implies

$$H(\pi,\lambda) = \int_0^{\pi} y'(t,\lambda)\, dt = y(\pi,\lambda) - y(0,\lambda)$$

Making use of the Prüfer substitution as suggested in (4.4) we obtain

$$v(x,\lambda) = x\lambda + \pi/3 \qquad r(x,\lambda) \equiv 1 \tag{4.16}$$

Since our problem satisfies H1-H6 and the conditions of Theorem 5.1, we can determine the necessary integers as follows

$\inf_{\lambda\epsilon L} v(\pi,\lambda) = \inf_{\lambda\epsilon L} [\pi\lambda + \pi/3] = \pi^2 + \pi/3 < 7\pi/2$ implies $m = 3$

$\sup_{\lambda\epsilon L} v(\pi,\lambda) = \sup_{\lambda\epsilon L} [\pi\lambda + \pi/3] = 2\pi^2 + \pi/3 > 13\pi/2$ implies $n = 6$

Thus, there exist three non-empty sets of eigenvalues T_0, T_1, T_2 for the specific problem. Moreover, the number of distinct eigenvalues is at least 2.

Now, we determine the values of $v(\pi,\lambda)$ at the points λ_0 and λ_3. Recall that λ_0 and λ_3 are the values such that $v(b,\lambda)$ is equal to $(2m + 1)\pi/2$ and $(2n + 1)\pi/2$, respectively, i.e.:

$$v(\pi, \lambda_0) = 7\pi/2 \qquad v(\pi, \lambda_3) = 13\pi/2$$

Thus, $\lambda_0 = 3.16$ and $\lambda_3 = 6.16$. Then the left side of (4.11) has the value 0.9659. At the values λ_1 and λ_2 the right side of (4.11) has the values 1.134 and -2.866, respectively. Thus condition (4.15) is satisfied on each of the intervals $[\lambda_0, \lambda_1]$, $[\lambda_1, \lambda_2]$, $[\lambda_2, \lambda_3]$. In fact, we can use (4.11) to determine these eigenvalues to be

$$3.824 \qquad 4.212 \qquad 5.824$$

Note that these are not the only eigenvalues for the problem. By considering subsequent intervals for λ we can find higher and higher eigenvalues.

VI. CONCLUSIONS

For certain second order linear differential equations with several types of boundary conditions, this thesis presents methods to determine the existence of eigenvalues. More importantly, it introduces a new theorem which incorporates several previous theorems as a special case.

The problem of G. D. Birkhoff does not appear to be very complicated, at first glance, since it contains no first order terms. The first hint of difficulty is the requirement that the problem be self-adjoint. Elegance is not an attribute of the proofs, but it is very interesting to note throughout Birkhoff's paper the dependence on the work of Sturm and the lack of familiarity with that of Hilbert.

However, Birkhoff did not formulate a method for determining the eigenvalues and eigenfunctions. The work of G. J. Haltiner when specialized to second order provides a method. Not only does he employ a newly defined adjoint, but he also presents the Green's function in a

novel way. The composition of this function is quite significant. In effect, he shows that the Green's function can be separated into the sum of two functions. The first is equivalent to considering the adjoint equation as an initial value problem. The second is a determinant whose elements are combinations of the eigenfunctions of the original equation and its adjoint. It would be interesting to pursue the question of whether the Green's function can be put in this form when the boundary conditions contain an integral term. Haltiner also obtains the known result that the eigenvalues of the two problems are the same.

Whyburn, influenced by Hilbert rather than Sturm, proves his results in a more elegant manner. Rather than work with a second order equation, he converts it to a first order system. This approach is pleasing because it gives the intuitive feeling that similar results may hold for equations of higher order. Saddled with an integral boundary condition, he proves his theorem by comparison to a problem whose solution is already known. In this sense, his method parallels Birkhoff's. Many problems which fit Birkhoff's hypotheses will also satisfy Whyburn's, but the inclusion is not complete.

The main achievement of this thesis is a generalization of the work of Etgen and Tefteller. Their work is an extension of that of Whyburn. They first consider a second order problem with boundary conditions that are more general than Whyburn's. Using a geometrical argument, they prove existence and oscillation theorems for the more general problem. However, their hypotheses do not allow the application of their theorem to Whyburn's problem. Instead they prove a separate theorem for Whyburn's problem. This author proves a new theorem which not only deals with boundary value problems that are more general than Etgen and Tefteller's,

but also has Whyburn's problem as a special case. Thus, with this new theorem we can handle three types of boundary value problems which before would have required three separate theorems. These three types are:

1. In equation (4.2b) $\gamma_i(\lambda) \equiv \delta_i(\lambda) \equiv 0$, $i = 1,2$. (Whyburn's problem)

2. In equation (4.2b) $0 < \gamma_1^2(\lambda) + \delta_1^2(\lambda) \leq 1$ and $\gamma_2(\lambda) \geq 1$. (Etgen and Tefteller's problem)

3. In equation (4.2b) $\gamma_2(\lambda) \geq [\gamma_1^2(\lambda) + \delta_1^2(\lambda)]^{\frac{1}{2}}$, provided, in each case, that the other hypotheses of the theorem are fulfilled.

The theorems of both Birkhoff and Whyburn depend on the principle of superposition. Thus their methods could not be applied to a non-linear boundary value problem. The method used by this author does not depend on linearity and thus may prove equally effective in dealing with the non-linear problem. (See Etgen and Tefteller [5].) Since it is well known that non-linear systems are vital in the study of physical systems, it would be a valuable contribution if a theorem similar to theorem 4.1 were developed for the non-linear case.

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13. ABSTRACT

In sophomore and junior level ordinary differential equations one studies the classical Sturm-Liouville boundary value problem, where the boundary conditions are of the separated type. It is well known that under very reasonable hypotheses this problem has a discrete set of non-trivial solutions for a discrete set of eigenvalues which are countably infinite and tend to infinity. It is the purpose of this thesis to study the question of whether similar results hold for problems when the boundary conditions are replaced by conditions of the non-separted type and also conditions where an integral is added. In doing so, we are able to generalize some recent results of Etgen and Tefteller.

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